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The gamma flexible Weibull distribution: Properties and Applications

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Abstract: A new gamma flexible Weibull distribution is introduced, which presents a bathtub-shaped hazard rate, and some of its properties are obtained. The parameters are estimated via maximum likelihood, and a simulation study is performed to examine the consistency of the estimates. The utility of the proposed model is shown using three real applications.

Keywords: Bathtub, Bimodal, COVID-19, Maximum likelihood, Moment, Quantile function

MSC: 33B05, 60E05, 62P99, 65C05

1 Introduction

Various phenomena that occur in the real world can be explained by statistical distributions. For a long time, many of the common distributions (Weibull, gamma, Burr XII, Gumbel) were sufficient for this purpose. However, with computer science development, more flexible distributions have become mandatory. One way to generate new families of distribution is through techniques to generalize existing ones. The main characteristic of these generalizations is the addition of more parameters to their baseline distributions, thus increasing their flexibility.

The Weibull distribution is widely used in many fields, but it is not suitable for bathtub-shaped or unimodal hazard rates. Thus, several models have been developed to extend this distribution and increase the modeling ability, such as those in (Mudholkar and Srivastava, 1993), (Xie and Lai, 1996), (Xie et al., 2002), (Lai et al., 2003), (Famoye et al., 2005), and (Cordeiro et al., 2010), among others.

Of the various modifications made to the Weibull distribution, the one of interest in this article is the flexible Weibull (FW) distribution (Bebbington et al., 2007) with shape parameters $\alpha, \beta > 0$, cumulative distribution function (cdf)

$$G(x; \alpha, \beta) = 1 - \exp\left(-e^{\alpha x - \frac{\beta}{x}}\right), \quad x > 0,$$

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and probability density function (pdf)

$$g(x;\alpha,\beta)=\left(\alpha+\frac{\beta}{x^2}\right)\mathrm{e}^{\alpha x-\frac{\beta}{x}}\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right).$$

For $\beta=0$ and $\alpha=\log(\lambda)$, the FW model reduces to the exponential, and then it can be regarded as a generalization of the Weibull (Bebbington et al., 2007).

There are several extensions of the FW distribution such as those reported by (El-Gohary et al., 2015), (El-Desouky et al., 2016), (Mustafa et al., 2016), (El-Damcese et al., 2016), (El-Desouky et al., 2017), and (Ahmad and Iqbal, 2017).

Zografos and Balakrishnan (2009) and Ristić and Balakrishnan (2012) defined the cdf of the gamma-G class for any parent cdf $G(x)=G(x;\boldsymbol{\theta})$ with parameter vector $\boldsymbol{\theta}$ of dimension p, by (for $x\in\mathbb{R}$)

$$F(x)=F(x;a,\boldsymbol{\theta})=\frac{\gamma(a,-\log[1-G(x)])}{\Gamma(a)}=\frac{1}{\Gamma(a)}\int_0^{-\log[1-G(x)]}t^{a-1}\mathrm{e}^{-t}\mathrm{d}t, \qquad (1)$$

where $a>0$ is a shape parameter, and $\Gamma(\cdot)$ is the gamma function. For $a=1$, Equation (1) reduces to the parent G cdf.

Recently, the gamma-G family has received considerable attention in works by (Nadarajah et al., 2015), (Alzaatreh et al., 2014), (Nadarajah et al., 2015), (Cordeiro et al., 2016), (Bourguignon and Cordeiro, 2016), (Iriarte et al., 2017), (Guerra et al., 2017), and (David et al., 2021), among others.

The article unfolds as follows: Section 2 defines the gamma-flexible Weibull (GFW) distribution and a linear representation for its density. The moments and generating function are reported in Section 3. Section 4 estimates the parameters by the maximum likelihood method and conducts a simulation study. Three real data sets are analyzed in Section 5 to show the utility of the new model. Finally, we draw some conclusions in Section 6.

2 The GFW model and its linear representation

A random variable X follows the GFW distribution, say $X\sim\mathrm{GFW}(a,\alpha,\beta)$, if its cdf and pdf (omitting parameters in the functions) are

$$F(x)=\frac{\gamma\left[a,\exp\left(\alpha x-\frac{\beta}{x}\right)\right]}{\Gamma(a)}=\frac{1}{\Gamma(a)}\int_0^{\exp\left(\alpha x-\frac{\beta}{x}\right)}t^{a-1}\mathrm{e}^{-t}\mathrm{d}t,\quad t>0, \qquad (2)$$

and

$$f(x)=\frac{\left(\alpha+\frac{\beta}{x^2}\right)\left(\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)^a\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)}{\Gamma(a)}, \qquad (3)$$

respectively.

The FW distribution was introduced in engineering, but it can be used in several fields. So, the GFW distribution can also be adopted in a similar manner.

The hazard rate function (hrf) of X follows from the last two expressions.

The GFW is identical to the FW distribution when $a=1$. The calculations in all sections were done using R software (R Core Team, 2020).

Figure 1 displays some plots of the density of X, which can be symmetric, right-symmetric, left-symmetric, or bimodal. Plots of the hrf of X are reported in Figure 2, which has increasing, decreasing, bathtub, and unimodal shapes.

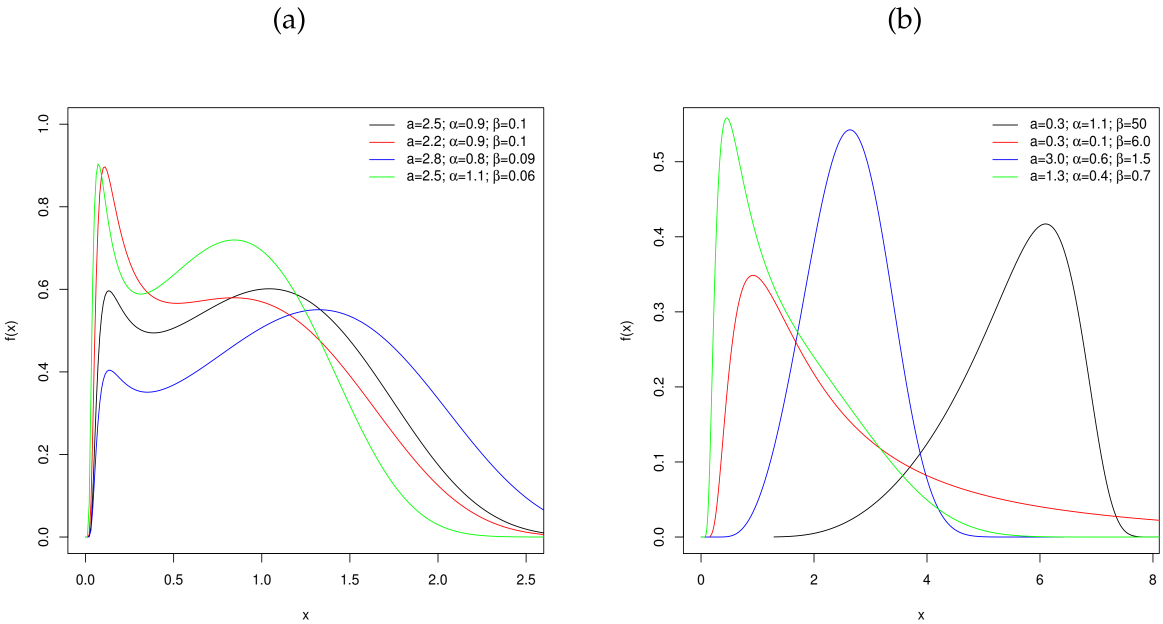

Figure 1: Plots of the density of X.

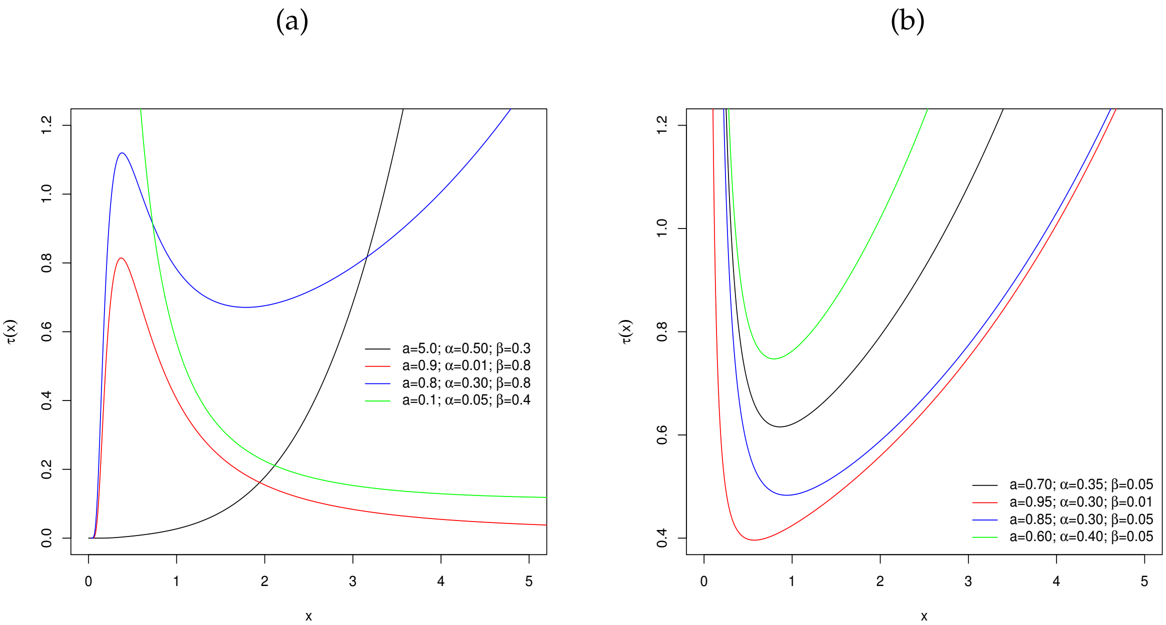

Figure 2: Plots of the hrf of X.

A simple motivation for the GFW distribution follows from Zografos and Balakrishnan (2009), where the GFW density can be approximated by the upper record value density from a sequence of independent and identically distributed FW random variables. Further, we highlight the utility of the proposed distribution in medical data analysis. In fact, the GFW distribution can be selected as the best model, especially in modeling unimodal and bimodal data of COVID-19 and cancer as illustrated in Section 5.

Following the concept of exponentiated distributions (Cordeiro et al., 2013), the exponentiated FW ("expFW") cdf with power parameter δ, say EFW(α,β,δ) (for $x>0$), is

$$H_\delta(x;\alpha,\beta)=\left[1-\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)\right]^{\delta}$$

and the corresponding pdf reduces to

$$h_\delta(x;\alpha,\beta)=\delta\left(\alpha+\frac{\beta}{x^2}\right)\mathrm{e}^{\alpha x-\frac{\beta}{x}}\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)\left[1-\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)\right]^{\delta-1}.$$

From Proposition 2 of Castellares and Lemonte (2015), we can write

$$[-\ln(1-v)]^c=v^c\sum_{m=0}^{\infty}\rho_m(c)\,v^m, \tag{4}$$

where $c\in\mathbb{R}$, $|v|<1$, $\rho_0(c)=1$, $\rho_m(c)=c\,\psi_{m-1}(m+c-1)$ for $m\geq 1$, and $\psi_m(\cdot)$ are Stirling polynomials, namely

$$\begin{aligned}\psi_{n-1}(w)=&\frac{(-1)^{n-1}}{(n+1)!}\left[T_n^{n-1}-\frac{w+2}{n+2}T_n^{n-2}+\frac{(w+2)(w+3)}{(n+2)(n+3)}T_n^{n-3}-\cdots\right.\\&\left.+(-1)^{n-1}\frac{(w+2)(w+3)\cdots(w+n)}{n+2)(n+3)\cdots(2n)}T_n^0\right],\end{aligned} \tag{5}$$

where $T_{n+1}^m=(2n+1-m)\,T_n^m+(n-m+1)\,T_n^{m-1}$ are positive integers, $T_0^0=1$, $T_{n+1}^0=1\times3\times5\times\cdots\times(2n+1)$, and $T_{n+1}^n=1$.

From Equation (4), we can rewrite Equation (3) as (Castellares and Lemonte, 2015)

$$f(x;a,\alpha,\beta)=\sum_{m=0}^{\infty}p_m\,h_{m+a}(x;\alpha,\beta)\,, \tag{6}$$

where $\varphi_0(a)=\Gamma(a)^{-1}$, $p_m=p_m(a)=\varphi_m(a)/(m+a)$, $\varphi_m(a)=\Gamma(a)^{-1}\rho_m(a-1)=(a-1)\,\Gamma(a)^{-1}\psi_{m-1}(m+a-2)$ (for $m\geq1$) can be determined from (5), and $h_{m+a}(x;\alpha,\beta)$ denotes the EFW density with power parameter $m+a$.

Equation (6) reveals that the GFW density is a linear combination of EFW densities. So, its properties can follow from those of the EFW distribution.

3 Moments and generating function

We calculate numerically in Table 1 the first four moments, standard deviation (SD), skewness (SK) and kurtosis (KR) of X varying a and β, with $\alpha=0.04$. The moments increase and the skewness and kurtosis decrease if β increases for a fixed. Note that the same happen when a increases for β fixed.

If $Y_{m+a}\sim$ EFW$(m+a,\alpha,\beta)$, we write from Equation (6)

$$\mu_r'=\mathbb{E}(X^r)=\sum_{m=0}^{\infty}p_m\,\mathbb{E}\left(Y_{m+a}^r\right)\,. \tag{7}$$

	$a = 0.1, \beta = 0.5$	$a = 0.1, \beta = 1.0$	$a = 0.1, \beta = 1.5$	$a = 0.1, \beta = 2.0$
μ'_1	0.458	0.6135	0.758	0.897
μ'_2	5.786	6.4989	7.263	8.0748
μ'_3	132.535	143.993	156.166	169.053
μ'_4	3639.751	3919.821	4215.782	4528.010
SD	2.361	2.474	2.586	2.696
SK	9.474	8.745	8.123	7.589
KR	109.462	95.521	84.187	74.898
	$a = 0.5, \beta = 0.5$	$a = 0.5, \beta = 1.0$	$a = 0.5, \beta = 1.5$	$a = 0.5, \beta = 2.0$
μ'_1	2.644	3.126	3.557	3.956
μ'_2	44.857	49.078	53.432	57.899
μ'_3	1108.423	1189.969	1275.122	1363.795
μ'_4	32176.930	34366.120	36656.030	39047.790
SD	6.153	6.269	6.385	6.500
SK	3.889	3.209	3.052	2.914
KR	15.476	14.290	13.283	12.420
	$a = 1.5, \beta = 0.5$	$a = 1.5, \beta = 1.0$	$a = 1.5, \beta = 1.5$	$a = 1.5, \beta = 2.0$
μ'_1	10.936	11.709	12.994	13.561
μ'_2	257.408	271.386	298.996	312.685
μ'_3	7526.057	7908.050	8692.924	9095.371
μ'_4	248547.900	261244.500	287697.100	301449.900
SD	11.738	11.588	11.408	11.347
SK	1.049	1.019	0.960	0.932
KR	3.218	3.200	3.143	3.113

Table 1: Numerical results for the GFW model.

Further, the rth moment of the EFW distribution is

$$\mathbb{E}(Y_{m+a}^r) = (m+a)\int_0^\infty x^r \left(\alpha + \frac{\beta}{x^2}\right) \mathrm{e}^{\alpha x - \frac{\beta}{x}} \exp\left(-\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right) \left[1 - \exp\left(-\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right)\right]^{m+a-1},$$

where $\left[1 - \exp\left(-\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right)\right]^{m+a-1}$ can be written as

$$\left[1 - \exp\left(-\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right)\right]^{m+a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(m+a)}{j!\,\Gamma(m+a-j)} \exp\left(-j\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right),$$

and then

$$\mathbb{E}(Y_{m+a}^r) = \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(m+a+1)}{j!\,\Gamma(m+a-j)} \int_0^\infty x^r \left(\alpha + \frac{\beta}{x^2}\right) \mathrm{e}^{\alpha x - \frac{\beta}{x}} \times \exp\left[-(j+1)\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right] \mathrm{d}x\,.$$

By using power series for $\exp\left[-(j+1)\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right]$ and $\mathrm{e}^{2(k+1)\alpha x}$ gives

$$\mathbb{E}(Y_{m+a}^r) = \sum_{j,k,i=0}^{\infty} \frac{(-1)^{j+k}\,(j+1)^k\,2^i\,(k+1)^i\,\Gamma(m+a+1)\alpha^i}{j!\,k!\,i!\,\Gamma(m+a-j)} \times \int_0^{\infty} x^{r+i}\left(\alpha+\frac{\beta}{x^2}\right)\mathrm{e}^{-(k+1)\alpha x-\frac{(k+1)\beta}{x}}\mathrm{d}x. \tag{8}$$

Based on the result (**3.471** 9) in Gradshteyn and Ryzhik (2007), we obtain

$$\mathbb{E}(Y_{m+a}^r) = \sum_{j,k,i=0}^{\infty} \frac{(-1)^{j+k}\,(j+1)^k\,2^i\,(k+1)^i\,\Gamma(m+a+1)\alpha^i}{j!\,k!\,i!\,\Gamma(m+a-j)} \times \left[2\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{\nu+1}{2}} K_{\nu+1}\left(2(k+1)\sqrt{\alpha\beta}\right) + 2\beta\left(\frac{\beta}{\alpha}\right)^{\frac{\nu-1}{2}} K_{\nu-1}\left(2(k+1)\sqrt{\alpha\beta}\right)\right], \tag{9}$$

where

$$\nu = r+i,\;\; K_\nu(z) = \frac{\pi\csc(\pi\nu)}{2}\left[I_{-\nu}(z)-I_\nu(z)\right]\;,\;\text{ and }\; I_\nu(z) = \sum_{\ell=0}^{\infty}\frac{1}{\Gamma(\ell+\nu+1)\ell!}\left(\frac{z}{2}\right)^{2\ell+\nu}$$

are the modified Bessel functions of the second and first kind, respectively (for $\nu \notin \mathbb{Z}$).

Substituting (9) into (7) gives the rth moment of the GFW distribution.

In a similar manner, the rth incomplete moment of X, say $m_r(s) = \int_0^s x^r\, f(x)\,\mathrm{d}x$, follows as

$$m_r(s) = \sum_{m,j,k,i=0}^{\infty} \frac{(-1)^{j+k}\,(j+1)^k\,2^i\,(k+1)^i\,p_m\,\Gamma(m+a+1)\alpha^i}{j!\,k!\,i!\,\Gamma(m+a-j)} \times \int_0^{s} x^{r+i}\left(\alpha+\frac{\beta}{x^2}\right)\mathrm{e}^{-(k+1)\alpha x-\frac{(k+1)\beta}{x}}\mathrm{d}x\,.$$

From Theorem 2 of Chaudhry and Zubair (1994), we obtain (for $r \geq 1$)

$$m_r(s) = \sum_{m,j,k,i=0}^{\infty} \frac{(-1)^{j+k}\,(j+1)^k\,2^i\,p_m\,\Gamma(m+a+1)}{j!\,k!\,i!\,(k+1)^r\,\Gamma(m+a-j)\,\alpha^r} \times \left\{\frac{\gamma\Big[(k+1)\alpha s;(r+i+1),(k+1)^2\alpha\beta\Big]}{(k+1)} + (k+1)\gamma\Big[(k+1)\alpha s;(r+i-1),(k+1)^2\alpha\beta\Big]\alpha\beta\right\},$$

where $\gamma(x;a,b) = \int_0^x t^{a-1}\,\mathrm{e}^{-t-b/t}\mathrm{d}t$ is the generalized lower incomplete gamma function.

The generating function (gf) of X can be written from (6) as

$$M(t) = \sum_{m=0}^{\infty} p_m\, M_{m+a}(t)\,, \tag{10}$$

where $M_{m+a}(t)$ is the gf of Y_{m+a}. The gf of the EFW distribution is

$$M_{m+a}(t) = (m+a)\int_0^\infty \mathrm{e}^{tx}\left(\alpha+\frac{\beta}{x^2}\right)\mathrm{e}^{\alpha x-\frac{\beta}{x}}\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)\left[1-\exp\left(-\mathrm{e}^{\alpha x-\frac{\beta}{x}}\right)\right]^{m+a-1}.$$

Following a similar algebra as for Equation (8) and again the result (**3.471** 9) (Gradshteyn and Ryzhik, 2007), we obtain (for $t<\alpha$)

$$\begin{aligned} M_{m+a}(t) = \sum_{j,k,i=0}^{\infty} & \frac{(-1)^{j+k}\,(j+1)^k\,2^i\,(k+1)^i\,\alpha^i\,\Gamma(m+a+1)}{j!\,k!\,i!\,\Gamma(m+a-j)} \\ & \times\left\{2\alpha\left[\frac{(k+1)\beta}{(k+1)\alpha-t}\right]^{\frac{i+1}{2}}K_{i+1}\left(2\sqrt{[(k+1)\alpha-t]\,(k+1)\,\beta}\right)\right. \\ & \left. +\,2\beta\left[\frac{(k+1)\beta)}{(k+1)\alpha-t}\right]^{\frac{i-1}{2}}K_{i-1}\left(2\sqrt{[(k+1)\alpha-t](k+1)\beta}\right)\right\}. \end{aligned} \tag{11}$$

Substituting Equation (11) into (10) gives the gf of the GFW distribution.

The quantile function (qf) of the FW distribution is given by (Bebbington et al., 2007)

$$Q_{\mathrm{FW}}(u;\alpha,\beta) = \frac{1}{2\alpha}\left\{\log\left[-\log(1-u)\right]+\sqrt{\{\log[-\log(1-u)]\}^2+4\alpha\beta}\right\}.$$

By inverting (2) and using results in Nadarajah et al. (2015), the qf of X follows as (for $0<u<1$)

$$Q_{\mathrm{GFW}}(u;a,\alpha,\beta) = \frac{1}{2\alpha}\left\{\log\{Q^{-1}[a,(1-u)]\}+\sqrt{\{\log[Q^{-1}(a,1-u)]\}^2+4\alpha\beta}\right\}, \tag{12}$$

where $Q^{-1}(a,u)$ is the inverse function of $Q(a,x)=1-\gamma(a,x)/\Gamma(a)$.

Approximations for the skweness and kurtosis of X can be based on quantile measures from (12). Let $Q_{\mathrm{GFW}}(u)=Q_{\mathrm{GFW}}(u;a,\alpha,\beta)$. The Bowley's skewness (Kenney and Keeping, 1962) is

$$\mathcal{B}(a,\alpha,\beta) = \frac{Q_{\mathrm{GFW}}(3/4)+Q_{\mathrm{GFW}}(1/4)-2Q_{\mathrm{GFW}}(1/2)}{Q_{\mathrm{GFW}}(3/4)-Q_{\mathrm{GFW}}(1/4)},$$

whereas the Moors kurtosis (Moors, 1988) is

$$\mathcal{M}(a,\alpha,\beta) = \frac{Q_{\mathrm{GFW}}(7/8)-Q_{\mathrm{GFW}}(5/8)-Q_{\mathrm{GFW}}(3/8)+Q_{\mathrm{GFW}}(1/8)}{Q_{\mathrm{GFW}}(6/8)-Q_{\mathrm{GFW}}(2/8)}.$$

Plots of these quantities for some choices of α and β as functions of a are reported in Figure 3. Note that the skewness increases when a goes to one and decreases from this value. The kurtosis decreases rapidly for small values of a and stabilizes when a increases.

An application of (12) using the first incomplete moment $m_1(s)$ refers to the Bonferroni and Lorenz curves defined by (for a given probability π)

$$B(\pi) = \frac{m_1(q)}{\pi\mu_1'} \quad \text{and} \quad L(\pi) = \frac{m_1(q)}{\mu_1'},$$

respectively, where $q=Q_{\mathrm{GFW}}(\pi)$. Plots of these curves versus π for some choices of a (with $\alpha=0.01$ and $\beta=15$) are displayed in Figure 4.

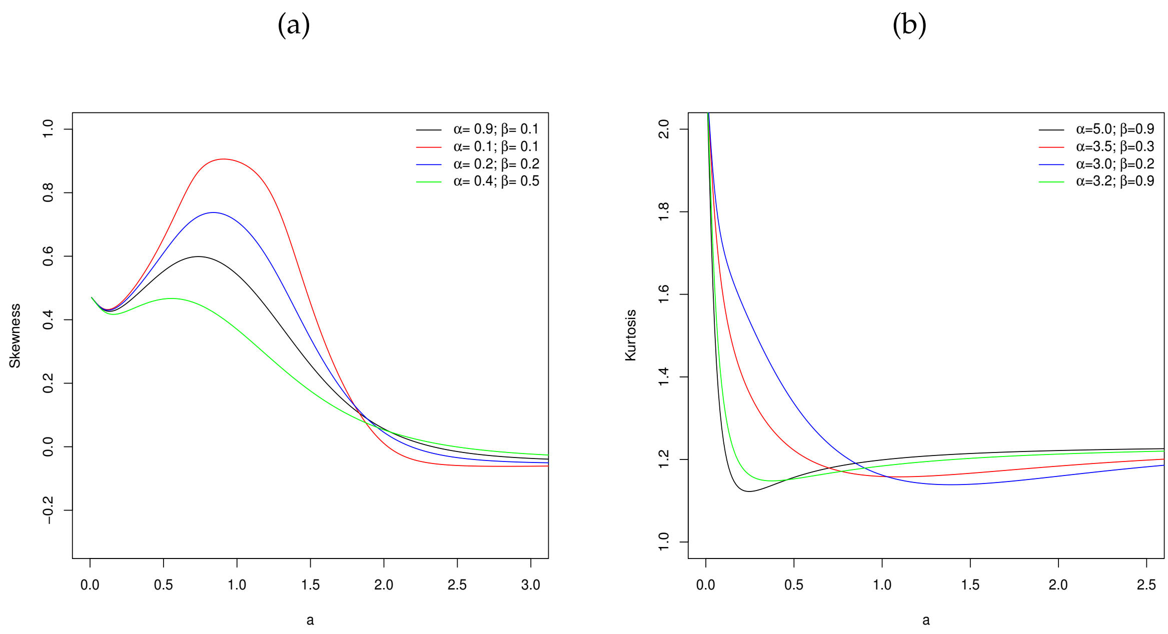

Figure 3: Skewness (a) and kurtosis (b) of X versus a.

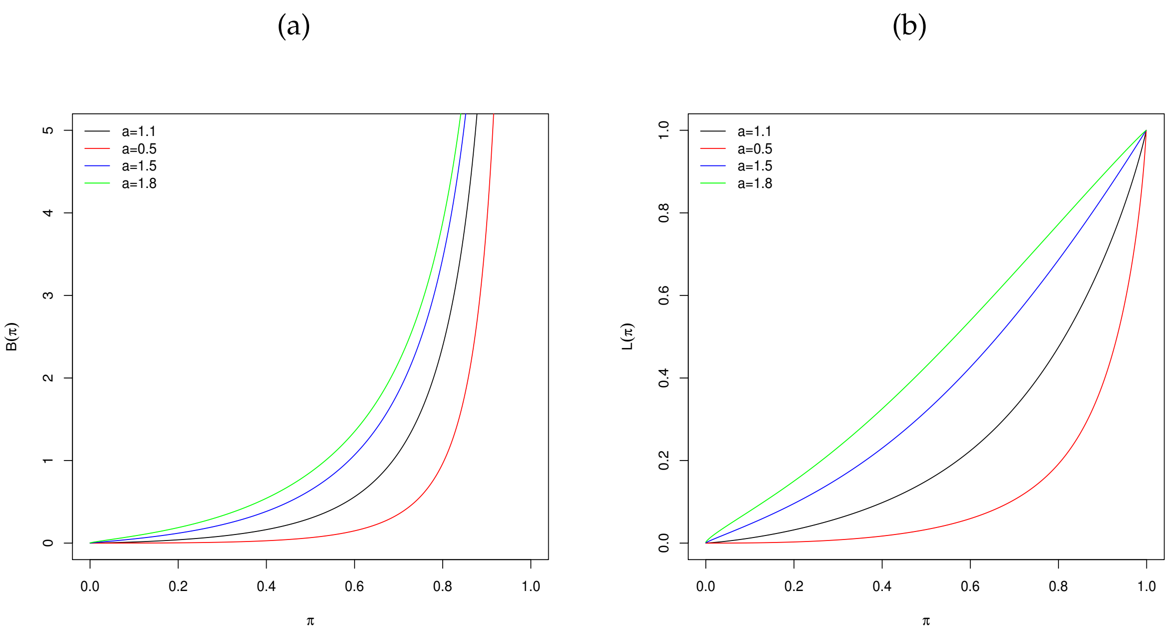

Figure 4: Bonferroni and Lorenz curves of X.

4 Estimation and Simulations

The log-likelihood function for $\boldsymbol{\theta} = (a, \alpha, \beta)^\top$ given the data set $x_1, \ldots, x_n$ from X is

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log\left(\alpha + \frac{\beta}{x_i^2}\right) + a \sum_{i=1}^{n}\left(\alpha x_i - \frac{\beta}{x_i}\right) + \sum_{i=1}^{n}\left(-\mathrm{e}^{\alpha x_i - \frac{\beta}{x_i}}\right) - n \log[\Gamma(a)]. \quad (13)$$

The maximum likelihood estimate (MLE) of $\boldsymbol{\theta}$, say $\widehat{\boldsymbol{\theta}}$, can be found by maximizing Equation (13) numerically using scripts such as `optim` or `nlm` in `R`, `MaxBFGS` in `Ox`, and `PROC NLMIXED` in `SAS`.

We generate $1,000$ Monte Carlo replicates for the GFW model from Equation (12) with sample sizes $n = 50, 100, 300,$ and 500 under three scenarios: $(a, \alpha, \beta) = (0.9, 2, 1.2)$ for scenario I; $(a, \alpha, \beta) = (0.5, 1.5, 3)$ for scenario II; and $(a, \alpha, \beta) = (1.5, 0.5, 0.8)$ for scenario III. We use the `optim` script of `R` to maximize (13). The averages, biases and mean square errors (MSEs) of the estimates are listed in Table 2. The averages tend to the true parameter values and the biases and MSEs converge to zero when n increases, which reveal that the MLEs are consistent.

		Scenario I			Scenario II			Scenario III		
n	parameter	Average	Bias	MSE	Average	Bias	MSE	Average	Bias	MSE
50	a	0.806	-0.094	0.261	0.630	0.130	0.246	1.480	-0.020	0.290
	α	2.815	0.815	4.787	2.099	0.599	2.782	0.514	0.014	0.004
	β	2.312	1.112	8.537	4.623	1.623	26.617	1.044	0.244	0.574
100	a	0.809	-0.090	0.210	0.588	0.088	0.166	1.484	-0.015	0.189
	α	2.430	0.430	0.815	1.826	0.326	0.836	0.509	0.009	0.002
	β	1.840	0.640	1.960	3.894	0.894	9.191	0.962	0.162	0.385
300	a	0.868	-0.031	0.137	0.556	0.056	0.085	1.497	-0.003	0.063
	α	2.156	0.156	0.117	1.583	0.083	0.084	0.502	0.002	0.001
	β	1.450	0.250	0.436	3.203	0.203	1.380	0.840	0.040	0.078
500	a	0.903	0.003	0.110	0.542	0.042	0.049	1.499	-0.001	0.035
	α	2.096	0.096	0.061	1.542	0.042	0.044	0.501	0.001	0.001
	β	1.343	0.143	0.265	3.083	0.083	0.784	0.819	0.019	0.029

Table 2: Simulation results for the GFW model.

5 Applications

We present three applications of the new model and compare it to other distributions: exponentiated Weibull (EW) (Mudholkar and Srivastava, 1993), modified Weibull (MW) (Lai et al., 2003), beta Weibull (BW) (Famoye et al., 2005), FW, Kumaraswamy Weibull (KwW) (Cordeiro et al., 2010), and Kumaraswamy Burr XII (KwBXII) (Paranaíba et al., 2013).

The best model is chosen based on Cramér-von Mises (W^*), Anderson-Darling (A^*), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The MLEs, standard errors (SEs), and the statistics are found using the **AdequacyModel** script (Marinho et al., 2019) of `R` software.

5.1 Failure times

The failure times of 50 components (per 1000h) are (Murthy et al., 2004): 0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574,0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816,7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080.

Table 3 gives some descriptive statistics. The mean is greater than the median, and then the data are right-skewed and leptokurtic.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
3.3430	1.4140	4.1395	17.1350	1.4167	4.0846	0.0360	15.0800

Table 3: Descriptive statistics for failure times.

Table 4 reports the MLEs and their SEs (in parentheses). The MW, EW, KwW, and BW models have higher SEs related to their estimates, whereas the GFW, FW, and KwBXII models have accurate estimates.

Table 5 indicates that the GFW model gives the best fit to the data since it has the lowest statistics among all models. The generalized likelihood ratio (GLR) test (Vuong, 1989) is used to compare the GFW model against the FW ($GLR = 3.911$), MW ($GLR = 3.503$), EW ($GLR = 3.455$), KwW ($GLR = 3.372$), KwBXII ($GLR = 3.452$), and BW ($GLR = 3.450$) models for a significance level of 5%. These results show that the GFW distribution provides the best fit to the current data.

The plots of the estimated densities and estimated survival functions for the most competitive models are shown in Figure 5. The GFW distribution provides the closest approximations to the histogram and empirical survival function, which shows its utility for real-life applications.

Model	MLEs (SEs)				
GFW (a, α, β)	1.362	0.109	0.126		
	(0.190)	(0.013)	(0.033)		
FW (α, β)	0.099	0.183			
	(0.012)	(0.034)			
MW (α, λ, β)	0.496	0.034	0.562		
	(0.099)	(0.025)	(0.098)		
EW (α, λ, β)	0.290	0.770	0.785		
	(0.681)	(0.990)	(1.546)		
KwW (a, b, α, β)	0.118	2.368	4.551	0.046	
	(0.024)	(1.555)	(0.099)	(0.025)	
KwBXII (a, b, c, k, s)	0.121	2.199	4.381	1.193	21.015
	(0.019)	(0.477)	(0.147)	(0.217)	(0.125)
BW (a, b, α, β)	0.708	0.703	0.412	0.819	
	(1.392)	(1.460)	(1.575)	(1.057)	

Table 4: Findings from the fitted models to failure times.

5.2 COVID-19

The numbers of deaths from COVID-19 in 83 Illinois counties in the United States through December 2021 are: 169, 13, 28, 91, 13, 107, 4, 41, 31, 89, 46, 57, 108, 146, 35, 30, 32, 156, 38, 52, 21, 113, 73, 59, 130, 93, 10, 40, 101, 25, 36, 16, 15, 95, 90, 101, 21, 150, 57, 32, 34, 127, 184, 38, 69, 115, 78, 121, 165, 24, 53, 58, 72, 15, 38, 108, 85, 104, 39, 110, 82, 16, 58, 7, 15, 7, 107, 67, 74, 14, 8, 56, 29, 124, 52, 19, 72, 30, 66, 34, 196, 201, 98. See https://data.world/associatedpress/johns-hopkins-coronavirus-case-tracker.

Model	W^*	A^*	AIC	CAIC	BIC	HQIC
GFW	**0.042**	**0.257**	**193.850**	**194.372**	**199.586**	**196.035**
FW	0.079	0.414	195.846	196.101	199.670	197.302
MW	0.130	0.850	208.727	209.249	214.463	210.912
EW	0.150	0.946	210.713	211.234	216.449	212.897
KwW	0.131	0.861	210.706	211.595	218.355	213.619
BW	0.149	0.942	212.696	213.585	220.344	215.608
KwBXII	1.132	0.870	213.086	214.450	222.646	216.726

Table 5: Adequacy measures for the models fitted to failure times.

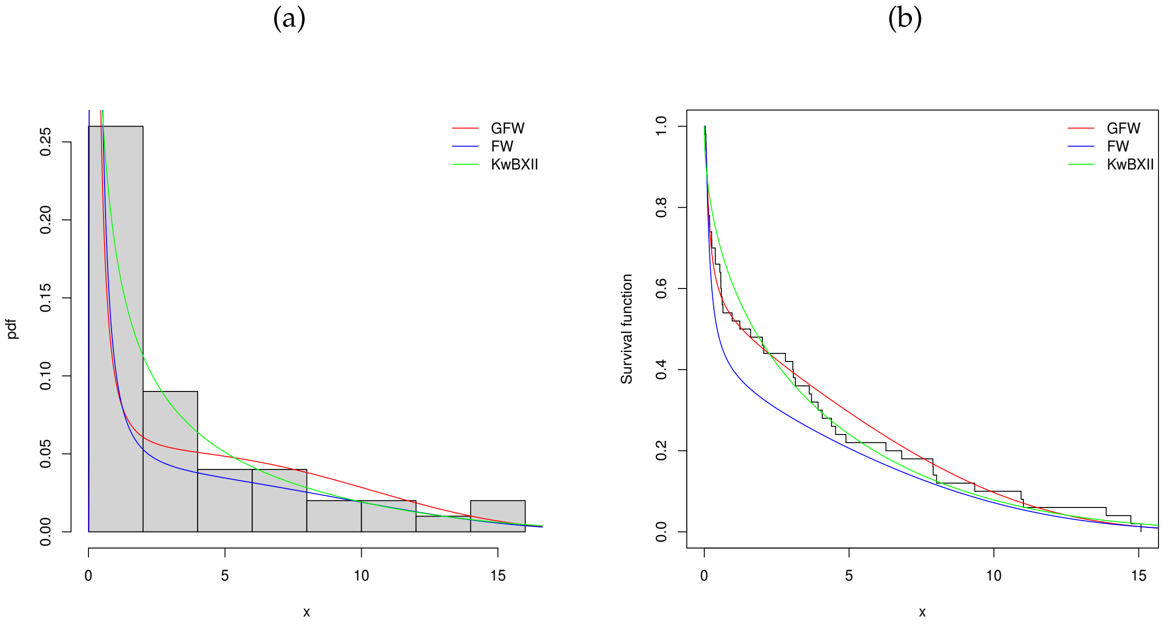

Figure 5: (a) Estimated densities of three models; (b) empirical and estimated survival functions of the models.

Table 6 shows some descriptive statistics for these data. The skewness is positive, and the kurtosis indicates mesokurtic distribution. The MLEs and their SEs (in parentheses) reported in Table 7 reveal

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
67.8670	57.0000	48.3850	2341.1	0.8198	2.9783	4	201

Table 6: Descriptive statistics for COVID-19 data.

that the GFW, FW, and KwW distributions have accurate estimates, and the other ones have high SEs relative to their estimates. The results in Table 8 indicate that the GFW model has the lowest values of the criteria, so it can be chosen as the best model. Additionally, the GLR test also reveals that the GFW model is better than the FW $(GLR = 3.383)$, MW $(GLR = 3.961)$, EW $(GLR = 2.925)$, KwW

($GLR = 2.473$), KwBXII ($GLR = 3.502$), and BW ($GLR = 4.698$) models for a significance level of 5%.

Figure 6 reports plots of the estimated densities and estimated cumulative functions for the most adequate models. The fit of the new distribution is closer to the histogram and empirical cumulative function than those of the other distributions. So, these results support that the GFW distribution is better suited to the current data.

Model	MLEs (SEs)				
GFW (a, α, β)	1.702	0.010	14.005		
	(0.253)	(0.001)	(4.458)		
FW (α, β)	0.008	32.812			
	(0.001)	(4.290)			
MW (α, β, λ)	0.005	0.003	1.161		
	(0.002)	(0.002)	(0.116)		
EW (α, β, λ)	0.013	1.418	0.986		
	(0.005)	(0.503)	(0.574)		
KwW (a, b, α, β)	1.333	0.117	1.336	0.069	
	(0.083)	(0.055)	(0.039)	(0.019)	
KwBXII (a, b, c, k, s)	10.526	72.271	0.327	1.393	40.836
	(25.224)	(95.623)	(0.401)	(2.142)	(122.510)
BW (a, b, α, β)	3.697	3.665	0.011	0.615	
	(1.303)	(1.943)	(0.006)	(0.120)	

Table 7: Findings from the fitted models to COVID-19 data.

Model	W^*	A^*	AIC	CAIC	BIC	HQIC
GFW	**0.034**	**0.241**	**855.333**	**855.637**	**862.590**	**858.248**
FW	0.095	0.596	859.516	859.666	864.353	861.459
MW	0.057	0.348	858.767	859.071	866.024	861.682
EW	0.059	0.351	858.690	858.994	865.946	861.605
KwW	0.056	0.335	860.250	860.763	869.925	864.137
BW	0.088	0.544	863.876	864.389	873.551	867.763
KwBXII	0.083	0.508	864.870	865.649	876.964	869.728

Table 8: Adequacy measures for the models fitted to COVID-19 data.

5.3 Laryngeal cancer

The data set corresponds to the lifetime (in months) of 90 male patients with laryngeal cancer. The data are (Colosimo and Giolo, 2006): 0.6, 1.3, 2.4, 3.2, 3.3, 3.5, 3.5, 4.0, 4.0, 4.3, 5.3, 6.0, 6.4, 6.5, 7.4, 2.5, 3.2, 3.3, 4.5, 4.5, 5.5, 5.9, 5.9, 6.1, 6.2, 6.5, 6.7, 7.0, 7.4, 8.1, 8.1, 9.6, 10.7, 0.2, 1.8, 2.0, 3.6, 4.0, 6.2, 7.0, 2.2, 2.6, 3.3, 3.6, 4.3, 4.3, 5.0, 7.5, 7.6, 9.3, 0.3, 0.3, 0.5, 0.7, 0.8, 1.0, 1.3, 1.6, 1.8, 1.9, 1.9, 3.2, 3.5, 5.0, 6.3, 6.4, 7.8, 3.7, 4.5, 4.8, 4.8, 5.0, 5.1, 6.5, 8.0, 9.3, 10.1, 0.1, 0.3, 0.4, 0.8, 0.8, 1.0, 1.5, 2.0, 2.3, 3.6, 3.8, 2.9, 4.3.

Some descriptive statistics in Table 9 reveal that the data are right-skewed and platykurtic. For these data, we compare the GFW distribution with other models that also have the bimodal shape,

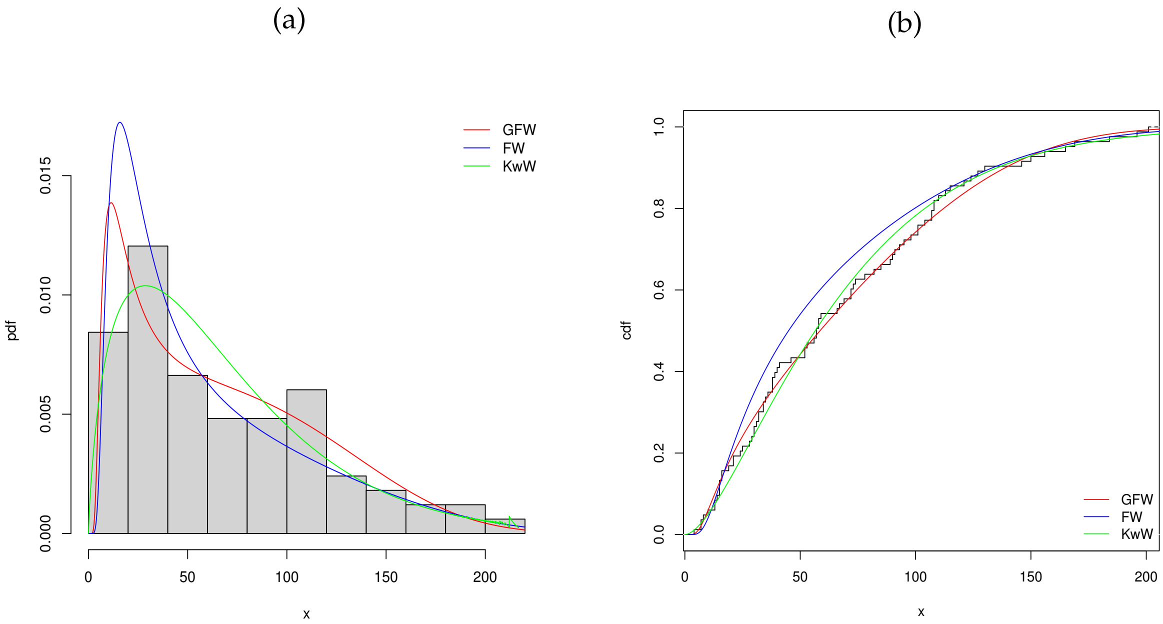

Figure 6: (a) Estimated densities of three models; (b) empirical and estimated cumulative functions of the models.

namely, the Odd log-logistic flexible Weibull (OLLFW) (Prataviera et al., 2018), extended Weibull log-logistic (EWLL) (Abouelmagd et al., 2019), Marshall-Olkin flexible Weibull (MOFW) (Mustafa et al., 2016), and FW.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
4.197	4	2.612	6.901	0.343	2.367	0.1	10.700

Table 9: Descriptive statistics for laryngeal cancer data.

The MLEs and their corresponding SEs (in parentheses) in Table (10) show that the GFW, OLLFW, MOFW, and FW distributions have accurate estimates. The GFW distribution has the lowest values of the adequacy measures in Table (11) and can provide a better fit than the other distributions. The GLR test confirms that the GFW distribution fits the current data better than the OLLFW $(GLR = 4.657)$, EWLL $(GLR = 3.741)$, MOFW $(GLR = 11.556)$, and FW $(GLR = 3.603)$ distributions for a significance level of 5%. The plots in Figure 7 also support our claim.

6 Conclusions

We introduced a new versatile distribution called the gamma flexible Weibull and provided some of its properties. A simulation study demonstrated that the maximum likelihood estimates of the parameters are consistent. Three real applications showed that the new distribution is extremely competitive to other lifetime models for unimodal and bimodal medical data.

Model	MLEs (SEs)		
GFW (a, α, β)	2.527	0.201	0.197
	(0.223)	(0.012)	(0.070)
OLLFW (a, α, β)	0.359	0.288	2.869
	(0.053)	(0.023)	(0.106)
EWLL (λ, α, β)	0.072	0.925	21.689
	(0.459)	(0.457)	(135.256)
MOFW (a, α, β)	6.118	0.188	0.453
	(1.528)	(0.011)	(0.142)
FW (α, β)	0.142	1.286	
	(0.012)	(0.198)	

Table 10: Findings from the fitted models to laryngeal cancer data.

Model	W^*	A^*	AIC	CAIC	BIC	HQIC
GFW	**0.035**	**0.211**	**414.251**	**414.530**	**421.751**	**417.275**
OLLFW	0.310	1.744	440.411	440.690	447.910	443.435
EWLL	0.148	0.913	425.014	425.293	432.514	428.038
MOFW	0.048	0.278	415.136	415.415	422.635	418.160
FW	0.564	3.218	462.138	462.275	467.137	464.154

Table 11: Adequacy measures for the models fitted to laryngeal cancer data.

(a) (b)

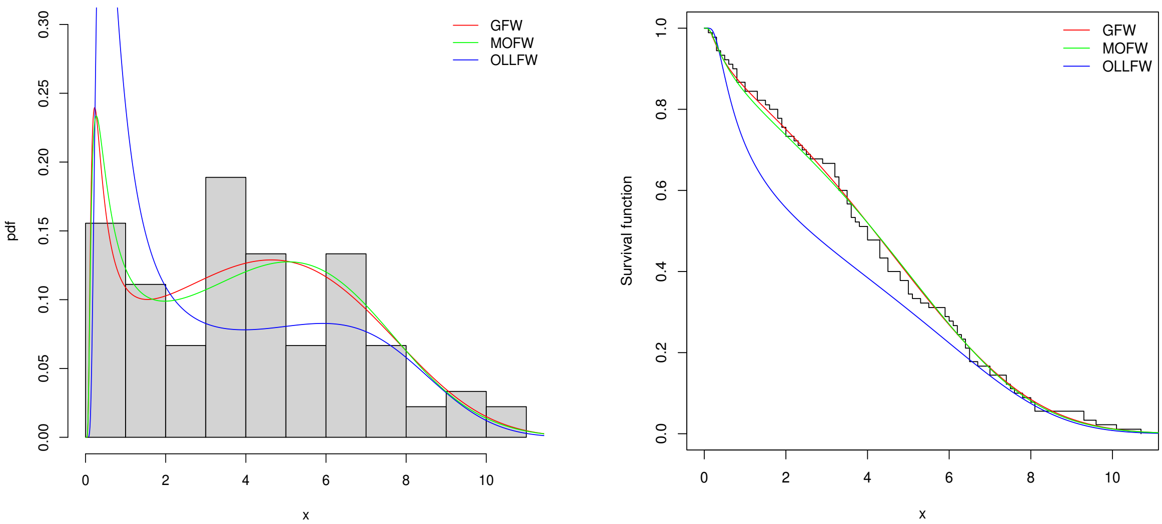

Figure 7: (a) Estimated densities of three models; (b) empirical and estimated survival functions of the models.

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